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Dynamical correlation functions of the spin Calogero–Sutherland model

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Abstract. The thermodynamic limit of the dynamical density and spin-density two-point correlation functions for the spin Calogero–Sutherland model are derived from Uglov’s finite-size results. The resultant formula for the density two-point correlation function is consistent with the previous conjecture on the basis of the minimal number of elementary excitations.

1. Introduction

In recent years, a considerable number of studies have been made on the $SU(N)$ spin Calogero–Sutherland (CS) model [1, 2]. This model describes an n particle system on a circle of length L interacting with the inverse-square type potential. Each particle is labelled by its coordinate x_j and spin with $N(\geq 1)$ possible values. (When $N = 1$, this is the CS model [3, 4].) The Hamiltonian of the model is given by

$$H = -\frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \left(\frac{\pi}{L}\right)^2 \sum_{1 \leq i < j \leq n} \frac{\beta(\beta + P_{ij})}{\sin^2 \frac{\pi}{L}(x_i - x_j)} \quad (1)$$

where β is a coupling parameter and P_{ij} is the spin exchange operator for particles i and j . In this paper, we take $N = 2$ and β to be a positive integer.

A lot of intriguing results have been obtained in connection with the spin CS model. In particular, the eigenfunctions of the spin CS model have been explicitly constructed and then, using these properties, the dynamical correlation functions of this model were computed. For $\beta = 1$ which is the simplest nontrivial case, the hole propagator of the $SU(2)$ spin CS model (with finite n and in the thermodynamic limit) has been calculated by Kato [5]. He also gave a conjectural formula for the arbitrary integer coupling case. Using the Jack polynomials with prescribed symmetry [6, 7], this conjecture was recently confirmed by Kato and one of the authors [8]. On the other hand, introducing the new class of orthogonal polynomials [9, 10], exact results have been obtained by Uglov [10]. He computed the dynamical density and spin-density two-point correlation functions of the $SU(2)$ spin CS model with finite n . In [11], Kato *et al* have studied the construction for the dynamical correlation functions of the $SU(N)$ spin CS model in the thermodynamic limit. We gave the formula for the density two-point correlation function in the thermodynamic limit and checked the consistency with the

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predictions from conformal field theory [12]. However, the relation between Uglov’s work and ours is missing.

In this paper, from Uglov’s formulae for the dynamical correlation function of the spin CS model, we take the thermodynamic limit which is technically nontrivial. For the density two-point correlation function, we prove our previous result in the thermodynamic limit microscopically.

Here, we give our main results together with those of the work in [5, 8]. We introduce the following notations: for non-negative integers a, b and c ,

$$\mathcal{E}(u, v, w; a, b, c) = \sum_{i=1}^a \epsilon_p(u_i) + \sum_{j=1}^b \epsilon_h(v_j) + \sum_{k=1}^c \epsilon_h(w_k) \tag{2}$$

$$\mathcal{P}(u, v, w; a, b, c) = \frac{\pi\rho_0}{2} \left[- (2\beta + 1) \sum_{i=1}^a u_i + \sum_{j=1}^b v_j + \sum_{k=1}^c w_k \right] \tag{3}$$

$$I(a, b, c)[*] = \prod_{i=1}^a \int_1^\infty du_i \prod_{j=1}^b \int_{-1}^1 dv_j \prod_{k=1}^c \int_{-1}^1 dw_k (*) |F_\beta(u, v, w; a, b, c)|^2 \tag{4}$$

where ρ_0 is the density of particles, variables $u = (u_1, \dots, u_a)$, $v = (v_1, \dots, v_b)$ and $w = (w_1, \dots, w_c)$ represent the normalized momenta of quasiparticle with spin σ , quasiholes with spin $-\sigma$ and σ , respectively ($\sigma = \pm \frac{1}{2}$). The quasiparticle and quasihole dispersions are introduced by

$$\epsilon_p(y) = (2\beta + 1)^2 \frac{1}{2} \left(\frac{\pi\rho_0}{2} \right)^2 (y^2 - 1) \tag{5}$$

$$\epsilon_h(y) = (2\beta + 1)^2 \frac{1}{2} \left(\frac{\pi\rho_0}{2} \right)^2 (1 - y^2) \tag{6}$$

respectively. The function F_β is defined by

$$\begin{aligned} F_\beta(u, v, w; a, b, c) = & \left(\prod_{1 \leq i < j \leq b} (v_i - v_j)^{g_h^d} \prod_{1 \leq i < j \leq c} (w_i - w_j)^{g_h^d} \prod_{i=1}^b \prod_{j=1}^c (v_i - w_j)^{g_h^0} \right) \\ & \times \left(\prod_{i=1}^a \prod_{j=1}^b (u_i - v_j) \prod_{i=1}^a (u_i^2 - 1)^{(1-g_p^d)/2} \right. \\ & \left. \times \prod_{j=1}^b (1 - v_j^2)^{(1-g_h^d)/2} \prod_{k=1}^c (1 - w_k^2)^{(1-g_h^d)/2} \right)^{-1} \end{aligned} \tag{7}$$

where

$$g_p^d = \beta + 1 \quad g_h^d = (\beta + 1)/(2\beta + 1) \quad g_h^0 = -\beta/(2\beta + 1). \tag{8}$$

The retarded Green function [5, 8], density and spin-density two-point correlation functions can respectively be written as the following form:

$$\langle \psi(x, t) \psi(0, 0) \rangle = A(\beta) I(0, \beta + 1, \beta) [(\pi\rho_0/2) e^{i(\mathcal{P}(0, \beta+1, \beta)x - (\mathcal{E}(0, \beta+1, \beta) - \zeta)t)}] \tag{9}$$

$$\langle \rho(x, t) \rho(0, 0) \rangle = B(\beta) I(1, \beta + 1, \beta) [\mathcal{P}(1, \beta + 1, \beta)^2 \cos(\mathcal{P}(1, \beta + 1, \beta)x) e^{-i\mathcal{E}(1, \beta+1, \beta)t}] \tag{10}$$

$$\begin{aligned} \langle s(x, t) s(0, 0) \rangle = & C_I(\beta) I(1, \beta, \beta + 1) [(\pi\rho_0/2)^2 \cos(\mathcal{P}(1, \beta, \beta + 1)x) e^{-i\mathcal{E}(1, \beta, \beta+1)t}] \\ & + C_{II}(\beta) I(1, \beta + 2, \beta - 1) [(\pi\rho_0/2)^2 \cos(\mathcal{P}(1, \beta + 2, \beta - 1)x) e^{-i\mathcal{E}(1, \beta+2, \beta-1)t}] \end{aligned} \tag{11}$$

where $\zeta = (\pi(2\beta + 1)\rho_0/2)^2$ is the chemical potential and we use the convention $\prod_{i=1}^0 (*)=1$. Constant factors in the above formulae are defined by

$$A(\beta) = \frac{1}{\pi(2\beta + 1)^\beta} D(\beta) \tag{12}$$

$$B(\beta) = \frac{1}{\pi^2(2\beta + 1)^{\beta+1}} D(\beta) \tag{13}$$

$$C_I(\beta) = \frac{1}{4\pi^2(2\beta + 1)^{\beta-1}} D(\beta) \tag{14}$$

$$C_{II}(\beta) = \frac{1}{4\pi^2(2\beta + 1)^{\beta-1}} \frac{\beta}{\beta + 2} D(\beta) \tag{15}$$

where, using the gamma function $\Gamma(z)$, the constant $D(\beta)$ is given by

$$D(\beta) = \frac{1}{\Gamma(\beta + 2)} \prod_{j=1}^{2\beta+1} \frac{\Gamma((\beta + 1)/(2\beta + 1))}{\Gamma^2(j/(2\beta + 1))}. \tag{16}$$

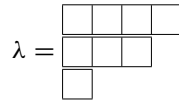
The paper is organized as follows. In section 2, we recall Uglov’s results about the dynamical correlation functions of the $SU(2)$ spin CS model. In section 3, first, we examine the excitation contents of the intermediate states of the dynamical correlation functions. Second, taking the thermodynamic limit, we derive the formulae (10) and (11). The conclusion is presented in section 4. Appendix A contains the proof of the statement in section 3.1. In appendix B, we give the examples of the explicit formulae of the building blocks for the dynamical correlation functions.

2. Uglov’s formulae for the dynamical correlation functions

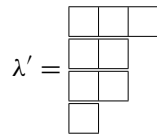
In this section, we fix notations and then recall Uglov’s exact results for the dynamical correlation functions [10]. We only give the final results of Uglov’s paper. For details, see [10].

2.1. Notations

In Uglov’s formulation, the states of the (transformed) Hamiltonian are labelled by the coloured partitions. A brief mathematical preliminary here may be in order. We fix notations which will be to the fore in this paper (see [13, 10]). For a fixed non-negative integer n , let $\Lambda_n = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in (\mathbb{Z}_{\geq 0})^n | \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$ be the set of all partitions with length less or equal to n . The weight of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is defined by $|\lambda| = \sum_{i=1}^n \lambda_i$. A partition can be represented by a Young diagram. For example, the partition $\lambda = (4, 3, 1)$ is expressed as



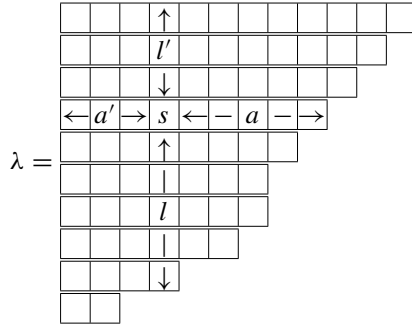
When there is a square in the i th row and j th column of λ , we write $(i, j) \in \lambda$. The conjugate of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$ whose diagram is the transpose of the diagram λ . For instance, if $\lambda = (4, 3, 1)$, then $\lambda' = (3, 2, 2, 1)$:



Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. For a square $s = (i, j) \in \lambda$, the numbers

$$\begin{aligned} a(s) &= \lambda_i - j & a'(s) &= j - 1 \\ l(s) &= \lambda'_j - i & l'(s) &= i - 1 \end{aligned} \tag{17}$$

are called arm-length, coarm-length, leg-length, and coleg-length, respectively:



Also the numbers

$$c(s) = a'(s) - l'(s) = j - i \tag{18}$$

$$h(s) = a(s) + l(s) + 1 = \lambda_i + \lambda'_j - i - j + 1 \tag{19}$$

are called content and hook-length, respectively. For $\alpha \in \mathbb{C}$, their refinements are defined by

$$c(s; \alpha) = a'(s) - \alpha l'(s) = j - 1 - \alpha(i - 1) \tag{20}$$

$$h_\lambda^*(s; \alpha) = a(s) + 1 + \alpha l(s) = \lambda_i - j + 1 + \alpha(\lambda'_j - i) \tag{21}$$

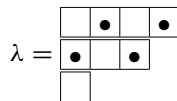
$$h_\lambda^\lambda(s; \alpha) = a(s) + \alpha(l(s) + 1) = \lambda_i - j + \alpha(\lambda'_j - i + 1). \tag{22}$$

Moreover we define the following numbers:

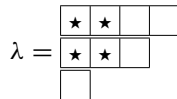
$$d(s; \alpha) = h_\lambda^*(s; \alpha)h_\lambda^\lambda(s; \alpha) \tag{23}$$

$$e(s; \alpha) = \frac{a'(s) + \alpha(n - l'(s))}{a'(s) + 1 + \alpha(n - l'(s) - 1)} = \frac{j - 1 + \alpha(n - i + 1)}{j + \alpha(n - i)}. \tag{24}$$

We recall a colouring scheme of diagrams. (Here we only need a colouring by two colours, white and black, since we consider the case with $N = 2$.) For a partition λ , we define two subsets of λ by $W_\lambda = \{s \in \lambda | c(s) \equiv 0 \pmod 2\}$ and $B_\lambda = \{s \in \lambda | c(s) \equiv 1 \pmod 2\}$. We call the colour of $s \in \lambda$ white (black) if $s \in W_\lambda$ ($\in B_\lambda$), and call $\lambda = W_\lambda \sqcup B_\lambda$ the coloured partition. (Notice that $(1, 1) \in W_\lambda$ (if $\lambda \neq \emptyset$.) For example, if $\lambda = (4, 3, 1)$, then $W_\lambda = \{\square$ in the following diagram}, and $B_\lambda = \{\bullet$ in the following diagram}:



We define an another subset of λ by $H_2(\lambda) = \{s \in \lambda | h(s) \equiv 0 \pmod 2\}$. For example, if $\lambda = (4, 3, 1)$, then $H_2(\lambda) = \{\star$ in the following diagram}:



2.2. Dynamical correlation functions

We now recall Uglov’s formulae for the dynamical correlation functions [10]. We take the number of particles n to be an even number such that $n/2$ is odd [10]. In Uglov’s formalism, the states of the CS Hamiltonian are labelled by the coloured partitions. First of all, the total energy with respect to the transformed Hamiltonian, total momentum, and total z -component of spin for the coloured partition λ are respectively given by†:

$$E_\lambda = \frac{1}{2} \left(\frac{2\pi}{L} \right)^2 \left[n_w(\lambda') - \gamma n_w(\lambda) + \frac{1}{2}((n-1)\gamma + 1)|W_\lambda| \right] \quad (25)$$

$$P_\lambda = \frac{2\pi}{L} |W_\lambda| \quad (26)$$

$$S_\lambda = |W_\lambda| - |B_\lambda| \quad (27)$$

where $\gamma = 2\beta + 1 (\in \mathbb{Z}_{>0})$, and

$$n_w(\lambda) = \sum_{s \in W_\lambda} l'(s) \quad (28)$$

$$n_w(\lambda') = \sum_{s \in W_\lambda} a'(s). \quad (29)$$

Here, for any subset $\mu \subset \lambda$, we denote by $|\mu|$ the number of squares in μ .

Next, the building blocks for the main factors of the dynamical correlation functions are defined as follows: for a coloured partition $\lambda \in \Lambda_n$,

$$X_\lambda = \prod_{s \in W_\lambda \setminus \{(1,1)\}} c(s; \gamma)^2 \quad (30)$$

$$Y_\lambda = \prod_{s \in H_2(\lambda)} d(s; \gamma) \quad (31)$$

$$Z_\lambda = \prod_{s \in W_\lambda} e(s; \gamma). \quad (32)$$

For the system with Hamiltonian (1), we denote the ground state expectation value of the operator \mathcal{O} by $\langle \mathcal{O} \rangle_n$. Then, the (ground state) dynamical density and spin-density two-point correlation functions are respectively given by [10]

$$\langle \rho(x, t) \rho(0, 0) \rangle_n = \frac{4}{\pi^2} \sum_{\substack{\lambda \in \Lambda_n: \text{coloured partition} \\ |\lambda|: \text{even} \\ S_\lambda = 0 \\ |W_\lambda| = |H_2(\lambda)|}} |P_\lambda|^2 X_\lambda Y_\lambda^{-1} Z_\lambda e^{-itE_\lambda} \cos(xP_\lambda) \quad (33)$$

$$\langle s(x, t) s(0, 0) \rangle_n = \frac{1}{2L^2} \sum_{\substack{\lambda \in \Lambda_n: \text{coloured partition} \\ |\lambda|: \text{odd} \\ S_\lambda = \pm 1 \\ |W_\lambda| = |H_2(\lambda)| + 1}} X_\lambda Y_\lambda^{-1} Z_\lambda e^{-itE_\lambda} \cos(xP_\lambda) \quad (34)$$

where $\rho(x, t)$ and $s(x, t)$ are the Heisenberg representations of the reduced density operator $\rho(x) = \sum_{i=1}^n \delta(x - x_i) - n/L$ and the z -component of spin-density operator $s(x) = \sum_{i=1}^n \delta(x - x_i) \sigma_i^z / 2$, respectively. Here σ_i^z is the z -component of Pauli matrices.

3. Thermodynamic limit of the dynamical correlation functions

In this section, we take the thermodynamic limit of Uglov’s exact formulae (33) and (34).

† The formula for E_λ in [10] has a typographical error.

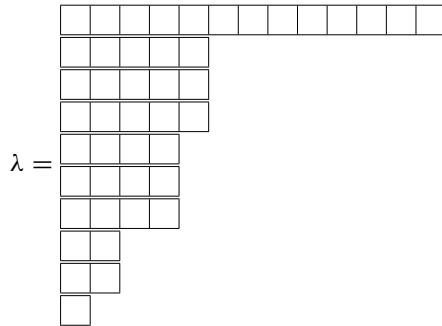
First, we determine the excitation contents of the intermediate states for the dynamical correlation functions (33) and (34). Second, we rewrite the formulae (33) and (34) in terms of parameters which correspond to the elementary excitations. Finally, we take the thermodynamic limit.

3.1. Intermediate states

In order to take the thermodynamic limit, we must determine the excitation contents of the intermediate states for the dynamical correlation functions (33) and (34).

Except for the factor X_λ which comes from the matrix element of the local operators, the factors in the sums of the right-hand side of (33) and (34) are non-zero for all λ . Therefore, a summand in the sums of the right-hand side of (33) and (34) is non-zero if and only if $X_\lambda \neq 0$. From the definition, it is easy to see that $X_\lambda \neq 0 \Leftrightarrow (2, \gamma + 1) \notin \lambda$ [10]. (Notice that $\gamma = 2\beta + 1 \in \mathbb{Z}_{>0}$.) As is the spinless case [14, 15], this condition has the following physical interpretation: the intermediate states contributing to the dynamical correlation functions (33) and (34) have one quasiparticle and γ quasihole excitations. We will see that these elementary excitations have the $SU(2)$ spin degrees of freedom. The above condition together with the conditions on the sums in the formulae (33) and (34) determine the intermediate states.

We parametrize the intermediate states of the correlation functions (33) and (34). For this purpose, we introduce some notations. In consideration of above observation, we define the subset of Λ_n by $\Lambda_n^{(\gamma)} = \{\lambda \in \Lambda_n | (2, \gamma + 1) \notin \lambda\}$. That is, a partition $\lambda \in \Lambda_n^{(\gamma)}$ has γ columns and one ‘arm’. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda_n^{(\gamma)}$, we introduce the notation $\lambda = (\lambda'_1, \lambda'_2, \dots, \lambda'_\gamma; r)$, which consists of γ columns and one ‘arm’. Here $r = \lambda_1 - \gamma$. (If $\lambda_1 < \gamma$, then $r = 0$.) For example, $\lambda = (13, 5, 5, 5, 4, 4, 4, 2, 2, 1) = (10, 9, 7, 7, 4; 8) \in \Lambda_n^{(5)}$:



Using these notations, we can state that the intermediate states for the density two-point correlation function (33) are coloured partitions $\lambda \in \Lambda_n^{(\gamma)}$ with even weight, $S_\lambda = 0$ and $|W_\lambda| = |H_2(\lambda)|$. We call these coloured partitions the d–d admissible. Similarly, the intermediate states for the spin-density two-point correlation function (34) are coloured partitions $\lambda \in \Lambda_n^{(\gamma)}$ with odd weight, $S_\lambda = \pm 1$ and $|W_\lambda| = |H_2(\lambda)| + 1$. We call these coloured partitions the s–s admissible.

The above conditions on the intermediate states of dynamical correlation functions are rather complicated. Technical difficulty in taking the thermodynamic limit comes from these complicated conditions. Then, in the following, we simplify the conditions for d–d and s–s admissible coloured partitions. For this purpose, we introduce more notations.

For $\nu = (\nu_1, \dots, \nu_{\gamma+1}) \in \{0, 1\}^{\times(\gamma+1)}$, we define two subsets $I_1(\nu)$ and $I_2(\nu)$ of $I = \{1, \dots, \gamma\}$ by

$$I_1(\nu) = \{j \in \{1, \dots, \gamma\} | v_j = \begin{cases} (1 + (-1)^j)/2, & \text{if } \nu_{\gamma+1} = 0 \\ (1 + (-1)^{j-1})/2, & \text{if } \nu_{\gamma+1} = 1 \end{cases} \} \quad (35)$$

$$I_2(v) = I \setminus I_1(v). \quad (36)$$

For example, if $v = \overbrace{(1, 1, \dots, 1)}^{\gamma+1 \text{ times}}$, then $I_1(v) = \{j \in I | j : \text{odd}\}$ and $I_2(v) = \{j \in I | j : \text{even}\}$. We introduce the function $\rho : \Lambda_n^{(\gamma)} \rightarrow \{0, 1\}^{\times(\gamma+1)}$ by

$$\rho(\lambda) = (\sigma(\lambda'_1), \dots, \sigma(\lambda'_\gamma), \sigma(r)) \quad \text{if } \lambda = \langle \lambda'_1, \dots, \lambda'_\gamma; r \rangle$$

where $\sigma(a) = 0$ (1) if a is even (odd). We call $\rho(\lambda)$ the parity of $\lambda \in \Lambda_n^{(\gamma)}$.

It can be easy to show that, for a coloured partition $\lambda \in \Lambda_n^{(\gamma)}$ with parity $\rho(\lambda) = (v_1, \dots, v_{\gamma+1})$,

$$S_\lambda = |W_\lambda| - |B_\lambda| = \sum_{j=1}^{\gamma+1} (-1)^{j-1} v_j. \quad (37)$$

Using these formulae, we have

$$S_\lambda = \begin{cases} 0 \\ \pm 1 \end{cases} \Leftrightarrow \#I_1(\rho(\lambda)) = \begin{cases} \beta + 1 \\ \beta \text{ or } \beta + 2. \end{cases} \quad (38)$$

Here, for a set A , $\#A$ denotes the number of elements. Moreover, we can show that, for $\lambda \in \Lambda_n$ with even (resp. odd) weight,

$$S_\lambda = 0 \text{ (resp. } \pm 1) \Leftrightarrow |W_\lambda| = |H_2(\lambda)| \quad (\text{resp. } |H_2(\lambda)| + 1). \quad (39)$$

The proof of the statement (39) is given in appendix A. The statements (38) and (39) are essential to taking the thermodynamic limit.

From the above statements, we see that a coloured partition $\lambda \in \Lambda_n^{(\gamma)}$ is the d–d admissible if and only if $|\lambda|$ is even and $\#I_1(\rho(\lambda)) = \beta + 1$. Similarly, a coloured partition $\lambda \in \Lambda_n^{(\gamma)}$ is the s–s admissible if and only if $|\lambda|$ is odd and $\#I_1(\rho(\lambda)) = \beta$ or $\beta + 2$. The s–s admissible coloured partitions are divided into two types which are characterized by $\#I_1(\rho(\lambda)) = \beta$ or $\beta + 2$. We call former type I and the latter type II. The $SU(2)$ spin degrees of freedom of the elementary excitations are assigned as follows. For an admissible coloured partition $\lambda = \langle \lambda'_1, \dots, \lambda'_\gamma; r \rangle \in \Lambda_n^{(\gamma)}$ with parity $\rho(\lambda) = (v_1, \dots, v_{\gamma+1})$, the spin of quasiparticle is $\frac{1}{2}$ (resp. $-\frac{1}{2}$) if $v_{\gamma+1} = 0$ (resp. 1). On the other hand, the spin of quasihole corresponding to λ'_j is $\frac{1}{2}$ (resp. $-\frac{1}{2}$) if $v_{\gamma+1} = 0$ and $j \in I_2(\rho(\lambda))$ or $v_{\gamma+1} = 1$ and $j \in I_1(\rho(\lambda))$ (resp. $v_{\gamma+1} = 0$ and $j \in I_1(\rho(\lambda))$ or $v_{\gamma+1} = 1$ and $j \in I_2(\rho(\lambda))$).

Now, we can determine the excitation contents of the intermediate states for the dynamical correlation functions. The excitation contents of the intermediate states for the dynamical density two-point correlation function is given by the following set of quasiparticle and quasiholes:

$$\begin{aligned} & \text{one quasiparticle with spin } \sigma \\ & \beta + 1 \text{ quasiholes with spin } -\sigma \\ & \beta \text{ quasiholes with spin } \sigma \end{aligned} \quad (40)$$

where $\sigma = \pm \frac{1}{2}$. This is consistent with the result in [11]. Similarly, the excitation contents of the intermediate states for the dynamical spin-density two-point correlation function is given by the following sets of quasiparticle and quasiholes:

$$\begin{aligned} & \text{one quasiparticle with spin } \sigma \\ & \beta \text{ quasiholes with spin } -\sigma \\ & \beta + 1 \text{ quasiholes with spin } \sigma \end{aligned} \quad (41)$$

and

$$\begin{aligned}
 &\text{one quasiparticle with spin } \sigma \\
 &\beta + 2 \text{ quasiholes with spin } -\sigma \\
 &\beta - 1 \text{ quasiholes with spin } \sigma.
 \end{aligned} \tag{42}$$

It is remarkable that two types of the set of the elementary excitations contribute to the dynamical spin-density two-point correlation function.

For later convenience, using (38) and (39), we rewrite (33) and (34) as

$$\begin{aligned}
 \langle \rho(x, t) \rho(0, 0) \rangle_n &= \frac{4}{\pi^2} \sum_{\substack{v=(v_1, \dots, v_{\gamma+1}) \in \{0, 1\}^{\times(\gamma+1)} \\ \#I_1(v) = \beta+1}} \\
 &\times \sum_{\substack{\lambda \in \Lambda_n^{(\gamma)}: \text{coloured partition} \\ |\lambda|: \text{even} \\ \rho(\lambda) = v}} |P_\lambda|^2 X_\lambda Y_\lambda^{-1} Z_\lambda e^{-itE_\lambda} \cos(xP_\lambda) \\
 \langle s(x, t) s(0, 0) \rangle_n &= \frac{1}{2L^2} \sum_{\substack{v=(v_1, \dots, v_{\gamma+1}) \in \{0, 1\}^{\times(\gamma+1)} \\ \#I_1(v) = \beta}} \sum_{\substack{\lambda \in \Lambda_n^{(\gamma)}: \text{coloured partition} \\ |\lambda|: \text{odd} \\ \rho(\lambda) = v}} X_\lambda Y_\lambda^{-1} Z_\lambda e^{-itE_\lambda} \cos(xP_\lambda) \\
 &+ \frac{1}{2L^2} \sum_{\substack{v=(v_1, \dots, v_{\gamma+1}) \in \{0, 1\}^{\times(\gamma+1)} \\ \#I_1(v) = \beta+2}} \sum_{\substack{\lambda \in \Lambda_n^{(\gamma)}: \text{coloured partition} \\ |\lambda|: \text{odd} \\ \rho(\lambda) = v}} X_\lambda Y_\lambda^{-1} Z_\lambda e^{-itE_\lambda} \cos(xP_\lambda).
 \end{aligned} \tag{43}$$

The first summation in the right-hand side of (43) is taken over $2_\gamma C_{(\gamma+1)/2}$ different parities, since

$$2_\gamma C_{(\gamma+1)/2} = \#\{v = (v_1, \dots, v_{\gamma+1}) \in \{0, 1\}^{\times(\gamma+1)} \mid \#I_1(v) = \beta + 1\}. \tag{45}$$

Similarly, the first summations of the first and second line in the right-hand side of (44) are respectively taken over $2_\gamma C_{(\gamma-1)/2}$ and $2_\gamma C_{(\gamma+3)/2}$ different parities.

3.2. Quasiparticle and quasihole description of the dynamical correlation functions

To take the thermodynamic limit, our next task is to rewrite the formulae (43) and (44) in terms of parameters which correspond to the momenta of quasiholes and quasiparticle (see [14, 15]). We have already introduced such parameters, i.e., $\lambda = \langle \lambda'_1, \lambda'_2, \dots, \lambda'_\gamma; r \rangle$. The quantities $\lambda'_1, \lambda'_2, \dots, \lambda'_\gamma$ and r are respectively related to the momenta of quasiholes and quasiparticle.

Although it can be possible to proceed the calculation by using the parameters $\lambda'_1, \lambda'_2, \dots, \lambda'_\gamma$ and r , it is appropriate to introduce new parameters as follows. We define the following numbers [10]:

$$w_i(\lambda) = \#\{s \in i\text{th row of } \lambda \mid s : \text{white}\} \tag{46}$$

$$w_j(\lambda') = \#\{s \in j\text{th column of } \lambda \mid s : \text{white}\}. \tag{47}$$

We note that, using these numbers, we have $|W_\lambda| = \sum_{i=1}^n w_i(\lambda) = \sum_{j=1}^{\lambda_1} w_j(\lambda')$. Then, instead of $\lambda'_1, \lambda'_2, \dots, \lambda'_\gamma$ and r , we adopt $w_1(\lambda'), \dots, w_\gamma(\lambda')$, and $p = w_1(\lambda) - \gamma$ as the parameters. These two sets of parameters are related by the formulae

$$\lambda'_j = \begin{cases} 2w_j(\lambda') - 1 & \text{if } j: \text{odd}, \lambda'_j: \text{odd} \\ 2w_j(\lambda') & \text{if } \lambda'_j: \text{even} \\ 2w_j(\lambda') + 1 & \text{if } j: \text{even}, \lambda'_j: \text{odd} \end{cases} \tag{48}$$

and

$$r = \begin{cases} 2p + 1 & \text{if } r: \text{ odd} \\ 2p & \text{if } r: \text{ even.} \end{cases} \tag{49}$$

Let us rewrite the formulae for the dynamical correlation functions (43) and (44) by using $w_1(\lambda'), \dots, w_\gamma(\lambda')$, and p . First of all, for $\lambda \in \Lambda_n^{(\gamma)}$, we have

$$|W_\lambda| = \sum_{j=1}^{\lambda_1} w_j(\lambda') = \sum_{j=1}^{\gamma} w_j(\lambda') + p \tag{50}$$

$$n_w(\lambda) = \sum_{s \in W_\lambda} l'(s) = \sum_{\substack{1 \leq j \leq \gamma \\ j: \text{ odd}}} (w_j(\lambda')^2 - w_j(\lambda')) + \sum_{\substack{2 \leq j \leq \gamma-1 \\ j: \text{ even}}} w_j(\lambda')^2 \tag{51}$$

$$n_w(\lambda') = \sum_{s \in W_\lambda} a'(s) = \sum_{j=1}^{\gamma} (j - 1)w_j(\lambda') + p(p + \gamma). \tag{52}$$

Then, from the definitions (25) and (26), we obtain the formulae for E_λ and P_λ .

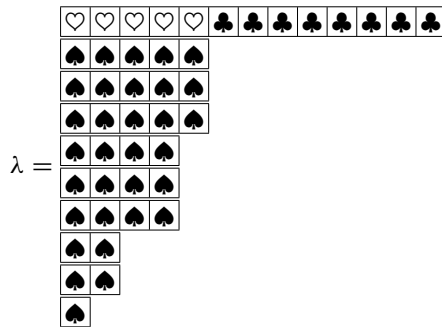
Next we rewrite X_λ, Y_λ and Z_λ . For this purpose, following Ha [14], we decompose a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda_n^{(\gamma)}$ into three sub-diagrams $\lambda = \mathcal{A}_\lambda \sqcup \mathcal{B}_\lambda \sqcup \mathcal{C}_\lambda$ where

$$\mathcal{A}_\lambda = \{(1, j) \in \Lambda_n^{(\gamma)} | 1 \leq j \leq \gamma\} \tag{53}$$

$$\mathcal{B}_\lambda = \{(i, j) \in \Lambda_n^{(\gamma)} | 1 \leq j \leq \gamma, 2 \leq i \leq \lambda'_j\} \tag{54}$$

$$\mathcal{C}_\lambda = \{(1, j) \in \Lambda_n^{(\gamma)} | \gamma + 1 \leq j \leq \lambda_1\}. \tag{55}$$

For example, if $\lambda = (13, 5, 5, 5, 4, 4, 4, 2, 2, 1) = \langle 10, 9, 7, 7, 4; 8 \rangle \in \Lambda_n^{(5)}$, then $\mathcal{A}_\lambda =$ sub-diagram which contains \heartsuit in the following diagram, $\mathcal{B}_\lambda =$ sub-diagram which contains \spadesuit , and $\mathcal{C}_\lambda =$ sub-diagram which contains \clubsuit :



We denote $W_{\mathcal{D}_\lambda} = W_\lambda \cap \mathcal{D}_\lambda$, $B_{\mathcal{D}_\lambda} = \mathcal{B}_\lambda \cap \mathcal{D}_\lambda$ and $H_{2, \mathcal{D}_\lambda} = H_2(\lambda) \cap \mathcal{D}_\lambda$ for $\mathcal{D} = \mathcal{A}, \mathcal{B}, \mathcal{C}$. (Notice that $(1, 1) \in W_{\mathcal{A}_\lambda}$ (if $\lambda \neq \emptyset$).

For a coloured partition $\lambda \in \Lambda_n^{(\gamma)}$, we denote $X_{\mathcal{D}_\lambda} = \prod_{s \in W_{\mathcal{D}_\lambda} \setminus \{(1,1)\}} c(s; \gamma)^2$ and $Y_{\mathcal{D}_\lambda} = \prod_{s \in H_{2, \mathcal{D}_\lambda}} d(s; \gamma)$ for $\mathcal{D} = \mathcal{A}, \mathcal{B}, \mathcal{C}$. It is obvious that $X_\lambda = X_{\mathcal{A}_\lambda} X_{\mathcal{B}_\lambda} X_{\mathcal{C}_\lambda}$, $Y_\lambda = Y_{\mathcal{A}_\lambda} Y_{\mathcal{B}_\lambda} Y_{\mathcal{C}_\lambda}$ and $Z_\lambda = Z_{\mathcal{A}_\lambda} Z_{\mathcal{B}_\lambda} Z_{\mathcal{C}_\lambda}$. Then, it is easy to show that, for $\lambda \in \Lambda_n^{(\gamma)}$,

$$X_{\mathcal{A}_\lambda} = 2^{\gamma-1} \Gamma^2((\gamma + 1)/2) \tag{56}$$

$$X_{\mathcal{B}_\lambda} = \xi^{\gamma+1} \prod_{j=1}^{\gamma} \xi^{-2w_j(\lambda')} \Gamma^{-2}(j/\gamma) \times \prod_{\substack{1 \leq j \leq \gamma \\ j: \text{ odd}}} \Gamma^2(w_j(\lambda') - \xi(j - 1)) \\ \times \prod_{\substack{2 \leq j \leq \gamma-1 \\ j: \text{ even}}} \Gamma^2(w_j(\lambda') + \frac{1}{2} - \xi(j - 1)) \tag{57}$$

$$X_{C_\lambda} = 2^{2p} \frac{\Gamma^2(p + (\gamma + 1)/2)}{\Gamma^2((\gamma + 1)/2)} \quad (58)$$

$$Y_{C_\lambda} = 2^{2p} \Gamma(p + 1) \frac{\Gamma(p + (\gamma + 1)/2)}{\Gamma((\gamma + 1)/2)} \quad (59)$$

$$Z_{A_\lambda} = \prod_{\substack{1 \leq j \leq \gamma \\ j: \text{odd}}} \frac{\gamma n + j - 1}{\gamma n + j - \gamma} \quad (60)$$

$$Z_{B_\lambda} = \prod_{\substack{1 \leq j \leq \gamma \\ j: \text{odd}}} \frac{\Gamma(n/2 + \xi j - \xi) \Gamma(n/2 - w_j(\lambda') + \xi j + 1/2)}{\Gamma(n/2 + \xi j - 1/2) \Gamma(n/2 - w_j(\lambda') + \xi j - \xi + 1)} \\ \times \prod_{\substack{2 \leq j \leq \gamma - 1 \\ j: \text{even}}} \frac{\Gamma(n/2 + \xi j - \xi + 1/2) \Gamma(n/2 - w_j(\lambda') + \xi j)}{\Gamma(n/2 + \xi j) \Gamma(n/2 - w_j(\lambda') + \xi j - \xi + 1/2)} \quad (61)$$

$$Z_{C_\lambda} = \frac{\Gamma(\gamma n/2 + 1) \Gamma(\gamma n/2 + p + (\gamma + 1)/2)}{\Gamma(\gamma n/2 + (\gamma + 1)/2) \Gamma(\gamma n/2 + p + 1)} \quad (62)$$

where $\xi = (2\gamma)^{-1}$. Notice that we can derive all above formulae without fixing the parity $\rho(\lambda)$.

On the other hand, to derive the explicit forms of Y_{A_λ} and Y_{B_λ} for $\lambda \in \Lambda_n^{(\gamma)}$, we must fix the parity $\rho(\lambda)$. In fact, to write down the explicit forms of Y_{A_λ} and Y_{B_λ} , we need more complicated notations. However, for the purpose of taking the thermodynamic limit, the necessary information are the sets $I_1(\rho(\lambda))$, $I_2(\rho(\lambda))$, and the quantities of order $\mathcal{O}(n)$. We see that, after replacing the elements of sets I_1 and I_2 appropriately, the thermodynamic limit of Y_{D_λ} and $Y_{D_{\lambda'}}$ with $\rho(\lambda) \neq \rho(\lambda')$ coincide with each other ($\mathcal{D} = \mathcal{A}, \mathcal{B}$). We do not give the explicit forms of Y_{A_λ} and Y_{B_λ} for the general admissible coloured partition λ . In appendix B, we give the examples for some admissible coloured partitions. The thermodynamic limit of Y_{A_λ} and Y_{B_λ} for general admissible coloured partitions are easily obtained from those examples.

Finally, we change the summation indices for the sums in the dynamical correlation functions. For example, we rewrite the sum in the density two-point correlation function (43) as

$$\sum_{\substack{v=(v_1, \dots, v_{\gamma+1}) \in \{0, 1\}^{\times(\gamma+1)} \\ \#I_1(v) = \beta + 1}} \sum_{p \geq 0} \sum_{n/2 \geq w_{j_1}(\lambda') \geq \dots \geq w_{j_{\beta+1}}(\lambda') \geq 0} \sum_{n/2 \geq w_{k_1}(\lambda') \geq \dots \geq w_{k_\beta}(\lambda') \geq 0} \quad (63)$$

where $\{j_l\}_{l=1}^{\beta+1} = I_1(\rho(\lambda) = v)$ such that $j_1 \geq \dots \geq j_{\beta+1}$ and $\{k_l\}_{l=1}^\beta = I_2(\rho(\lambda) = v)$ with $k_1 \geq \dots \geq k_\beta$.

3.3. Thermodynamic limit

In this section, we take the thermodynamic limit, i.e., $n \rightarrow \infty$, $L \rightarrow \infty$ with $\rho_0 = n/L$ fixed.

Let us introduce the momenta u and v_j for $j = 1, \dots, \gamma$ of the quasiparticle and quasiholes, respectively, by the formulae,

$$\frac{1}{\gamma} \frac{p}{n} \rightarrow -\frac{u + 1}{4} \quad (64)$$

$$\frac{w_j(\lambda')}{n} \rightarrow \frac{v_j + 1}{4}. \quad (65)$$

Then we have the thermodynamic limit of the energy and total momentum,

$$E_\lambda \rightarrow \mathcal{E} = \sum_{j=1}^{\gamma} \epsilon_h(v_j) + \epsilon_p(u) \quad (66)$$

$$P_\lambda \longrightarrow \mathcal{P} = \frac{\pi\rho_0}{2} \left[\sum_{j=1}^\gamma v_j - \gamma u \right] \tag{67}$$

where

$$\epsilon_h(y) = \gamma \frac{1}{2} \left(\frac{\pi\rho_0}{2} \right)^2 (1 - y^2) \tag{68}$$

$$\epsilon_p(y) = \gamma^2 \frac{1}{2} \left(\frac{\pi\rho_0}{2} \right)^2 (y^2 - 1). \tag{69}$$

We have adopted the normalization (64) and (65) of u and v_j so that the Fermi points coincide with $\{\pm 1\}$.

Also, using the formula $\lim_{|a| \rightarrow \infty} \Gamma(a+z)/\Gamma(a) = a^z$, we can obtain the thermodynamic limit of $X_\lambda Y_\lambda^{-1} Z_\lambda$. In the following, we consider the case of density two-point correlation function. In this case, we have

$$\begin{aligned} X_\lambda Y_\lambda^{-1} Z_\lambda &\longrightarrow L^{-(\gamma+1)} 2^{2\gamma} (\gamma\rho_0)^{-(\gamma+1)} \Gamma((\gamma+1)/2) \prod_{j=1}^\gamma \Gamma(\xi + \frac{1}{2}) \Gamma^{-2}(j/\gamma) \\ &\quad \times (u^2 - 1)^{(\gamma-1)/2} \prod_{j=1}^\gamma (1 - v_j^2)^{\xi-1/2} \prod_{j \in I_1(\rho(\lambda))} (u - v_j)^{-2} \\ &\quad \times \prod_{s=1,2} \prod_{\substack{j,k \in I_s(\rho(\lambda)) \\ j < k}} (v_j - v_k)^{-2(\xi+1/2)} \prod_{j \in I_1(\rho(\lambda))} \prod_{k \in I_2(\rho(\lambda))} (v_j - v_k)^{-2(\xi-1/2)} \end{aligned} \tag{70}$$

for a d–d admissible coloured partition $\lambda \in \Lambda_n^{(\gamma)}$ with the fixed parity $\rho(\lambda)$.

For each d–d admissible coloured partition $\lambda \in \Lambda_n^{(\gamma)}$ with the fixed parity $\rho(\lambda)$, we replace $\{v_j\}_{j \in I_1(\rho(\lambda))}$ and $\{v_j\}_{j \in I_2(\rho(\lambda))}$ by $\{v_j\}_{j=1}^{\beta+1}$ such that $v_1 \geq \dots \geq v_{\beta+1}$ and $\{w_j\}_{j=1}^\beta$ such that $w_1 \geq \dots \geq w_\beta$, respectively.

In the thermodynamic limit, we rewrite the sums as integrals:

$$\begin{aligned} \sum_{n/2 \geq w_{j_1}(\lambda') \geq \dots \geq w_{j_{\beta+1}}(\lambda') \geq 0} \sum_{n/2 \geq w_{k_1}(\lambda') \geq \dots \geq w_{k_\beta}(\lambda') \geq 0} &\longrightarrow L^\gamma 2^{-2\gamma} \rho_0^\gamma \\ &\times \int_{1 \geq v_1 \geq v_2 \geq \dots \geq v_{\beta+1} \geq -1} dv_1 dv_2 \dots dv_{\beta+1} \int_{1 \geq w_1 \geq w_2 \geq \dots \geq w_\beta \geq -1} dw_1 dw_2 \dots dw_\beta \end{aligned} \tag{71}$$

$$\sum_{p \geq 0} \longrightarrow -L 2^{-2} \gamma \rho_0 \int_{-\infty}^{-1} du. \tag{72}$$

Note that, for each d–d admissible coloured partition $\lambda \in \Lambda_n^{(\gamma)}$ with the fixed parity $\rho(\lambda)$, the energy \mathcal{E} , total momentum \mathcal{P} and thermodynamic limit of the quantity $X_\lambda Y_\lambda^{-1} Z_\lambda$ are invariant under the exchange $v_i \leftrightarrow v_j$ and/or $w_k \leftrightarrow w_l$. Then, finally, after removing the order on momenta, we arrive at the formula (10). This formula coincides with our previous result in [11] up to the constant factor.

The formula (11) can be derived in the same way.

The essential part of the formulae (9)–(11) can be described by the function F_β . As is the spinless case [14], we call the function F_β the minimal form factor of the $SU(2)$ spin CS model (with integer coupling parameter). The physical interpretation of the minimal form factor has been discussed in [11].

4. Conclusion

In this paper, we have taken the thermodynamic limit of dynamical density and spin-density two-point correlation functions of the spin CS model. We have obtained the exact formulae (10) and (11) of the density and spin-density two-point correlation functions, respectively. We have exactly shown that, with appropriate numbers of quasiparticles and quasiholes, the dynamical correlation functions of the spin CS model can be described by the unique function $F_\beta(7)$ which is called the minimal form factor. These results are consistent with our previous paper [11].

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Appendix A

In this appendix, we prove the following lemma.

Lemma. For a coloured partition $\lambda \in \Lambda_n$ with even (resp. odd) weight, $S_\lambda = 0$ (resp. ± 1) $\Leftrightarrow |W_\lambda| = |H_2(\lambda)|$ (resp. $|H_2(\lambda)| + 1$).

In this appendix, we do not assume that n is even.

We introduce the notations. The partition $\lambda = (\lambda_1, \lambda_2, \dots)$ can be represented by the notation $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots)$ where $m_i(\lambda) = \#\{j | \lambda_j = i\}$ [13]. Using this notation, we define the following transformations τ_i and τ'_i for $i \in \mathbb{Z}_{>0}$:

$$\tau_i : \lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots i^{m_i(\lambda)} \dots) \longrightarrow \begin{cases} (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots i^{m_i(\lambda)-2} \dots) & m_i(\lambda) \geq 2 \\ \lambda & m_i(\lambda) < 2 \end{cases} \quad (\text{A.1})$$

$$\tau'_i : \lambda \longrightarrow (\tau_i(\lambda'))'. \quad (\text{A.2})$$

That is, τ_i (τ'_i) is the following transformation: if there exist two rows (columns) which have the same number of squares i then τ_i (τ'_i) removes these rows (columns), if not then τ_i (τ'_i) is the identity. If λ has even (odd) weight then both $\tau_i(\lambda)$ and $\tau'_i(\lambda)$ have even (odd) weights. We introduce the special partition $\delta(k) \in \Lambda_n$ by

$$\delta(k) = \begin{cases} \emptyset & k = 0 \\ (k, k - 1, \dots, 1) = (1^1 2^1 \dots k^1) & k = 1, \dots, n. \end{cases} \quad (\text{A.3})$$

The partitions $\delta(k)$ for $k = 0, 1, \dots, n$ are the fixed points of the transformations τ_i and τ'_i . We see that, applying τ_i 's and τ'_i 's sufficiently many times, any partition $\lambda \in \Lambda_n$ is mapped to one of $\delta(k)$'s. We denote the resultant mapping by $\tau : \Lambda_n \longrightarrow \{\delta(k) | k = 0, 1, \dots, n\}$.

From the definition, the transformations τ_i , τ'_i and τ can be defined on the set of all coloured partitions. For instance,

$$\tau \left(\begin{array}{cccc} \square & \bullet & \square & \bullet \\ \bullet & \square & \bullet & \square \\ \square & \bullet & \square & \square \\ \bullet & \square & \square & \square \\ \square & \bullet & \square & \square \\ \bullet & \square & \square & \square \end{array} \right) = \begin{array}{cc} \square & \bullet \\ \bullet & \square \end{array} = \delta(2).$$

We define the following numbers: for a coloured partition $\lambda \in \Lambda_n$, $wb(\lambda) = |W_\lambda| - |B_\lambda| (= S_\lambda)$ and $wh(\lambda) = |W_\lambda| - |H_2(\lambda)|$. It is easy to see that these numbers are invariant under the transformations τ_i and τ'_i , i.e., $wb(\tau_i(\lambda)) = wb(\lambda)$, $wb(\tau'_i(\lambda)) = wb(\lambda)$ and same formulae for wh . Therefore, $wb(\tau(\lambda)) = wb(\lambda)$ and $wh(\tau(\lambda)) = wh(\lambda)$. Moreover, we have:

$$wb(\delta(k)) = \begin{cases} 0 & k = 0 \\ l & k = 2l - 1 \quad (l = 1, 2, \dots) \\ -l & k = 2l \quad (l = 1, 2, \dots) \end{cases} \tag{A.4}$$

$$wh(\delta(k)) = \begin{cases} 0 & k = 0 \\ l^2 & k = 2l - 1 \quad (l = 1, 2, \dots) \\ l^2 & k = 2l \quad (l = 1, 2, \dots). \end{cases} \tag{A.5}$$

(Note that $H_2(\delta(k)) = \emptyset$ for all k .)

We define the subset $\Lambda_n(\delta(k))$ of Λ_n by

$$\Lambda_n(\delta(k)) = \{\lambda \in \Lambda_n \mid \tau(\lambda) = \delta(k)\}. \tag{A.6}$$

It is important to note the following fact: if $\lambda \in \Lambda_n(\delta(k))$ then $wb(\lambda) = wb(\delta(k))$ and $wh(\lambda) = wh(\delta(k))$. Therefore, from the formulae (A.4) and (A.5), $\Lambda_n(\delta(k)) \cap \Lambda_n(\delta(k')) = \emptyset$ if $k \neq k'$. This fact proves the lemma.

We have the following decomposition for the set of all coloured partitions Λ_n :

$$\Lambda_n = \sqcup_{k=0}^n \Lambda_n(\delta(k)). \tag{A.7}$$

We see that the set of all d–d (resp. s–s) admissible coloured partitions is the set $\Lambda_n(\delta(0)) \cap \Lambda_n^{(\gamma)}$ (resp. $(\Lambda_n(\delta(1)) \sqcup \Lambda_n(\delta(2))) \cap \Lambda_n^{(\gamma)}$).

Appendix B

In this appendix, we give examples of the explicit formula for Y_{A_λ} and Y_{B_λ} .

(a) Example for the d–d admissible coloured partition.

We consider the d–d admissible coloured partition $\lambda \in \Lambda_n^{(\gamma)}$ with parity $\rho(\lambda) = \overbrace{(1, 1, \dots, 1)}^{\gamma+1 \text{ times}}$. In this case, $I_1(\rho(\lambda)) = \{j \in I \mid j : \text{odd}\}$ and $I_2(\rho(\lambda)) = \{j \in I \mid j : \text{even}\}$. We have the explicit formula for Y_{A_λ} and Y_{B_λ}

$$Y_{A_\lambda} = \xi^{-(\gamma+1)} \prod_{j \in I_1(\rho(\lambda))} (p/\gamma + w_j(\lambda') - \frac{1}{2} - \xi(j-2))(p/\gamma + w_j(\lambda') - \xi(j-1)) \tag{B.1}$$

$$\begin{aligned} Y_{B_\lambda} &= \xi^{\gamma+1} \Gamma((1 + 1/\gamma)/2)^{-\gamma} \prod_{j=1}^{\gamma} \xi^{-2w_j(\lambda')} \\ &\times \prod_{j \in I_1(\rho(\lambda))} \Gamma(w_j(\lambda') - \xi(j-1)) \Gamma(w_j(\lambda') - \xi j + \frac{1}{2}) \\ &\times \prod_{j \in I_2(\rho(\lambda))} \Gamma(w_j(\lambda') - \xi(j-1) + \frac{1}{2}) \Gamma(w_j(\lambda') - \xi j + 1) \\ &\times \prod_{\substack{j, k \in I_1(\rho(\lambda)) \\ j < k}} [\Gamma(w_j(\lambda') - w_k(\lambda') + \xi(k-j)) \Gamma(w_j(\lambda') \\ &- w_k(\lambda') + \xi(k-j) - \xi + \frac{1}{2})][\Gamma(w_j(\lambda') - w_k(\lambda') \\ &+ \xi(k-j) + \xi + \frac{1}{2}) \Gamma(w_j(\lambda') - w_k(\lambda') + \xi(k-j) + 1)]^{-1} \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{\substack{j,k \in I_2(\rho(\lambda)) \\ j < k}} [\Gamma(w_j(\lambda') - w_k(\lambda') + \xi(k - j))\Gamma(w_j(\lambda') \\
 & - w_k(\lambda') + \xi(k - j) - \xi + \frac{1}{2})][\Gamma(w_j(\lambda') - w_k(\lambda') \\
 & + \xi(k - j) + \xi + \frac{1}{2})\Gamma(w_j(\lambda') - w_k(\lambda') + \xi(k - j) + 1)]^{-1} \\
 & \times \prod_{\substack{j \in I_1(\rho(\lambda)), k \in I_2(\rho(\lambda)) \\ j < k}} [\Gamma(w_j(\lambda') - w_k(\lambda') + \xi(k - j) - \frac{1}{2})\Gamma(w_j(\lambda') \\
 & - w_k(\lambda') + \xi(k - j) - \xi)][\Gamma(w_j(\lambda') - w_k(\lambda') \\
 & + \xi(k - j) + \xi - 1)\Gamma(w_j(\lambda') - w_k(\lambda') + \xi(k - j) - \frac{1}{2})]^{-1} \\
 & \times \prod_{\substack{j \in I_2(\rho(\lambda)), k \in I_1(\rho(\lambda)) \\ j < k}} [\Gamma(w_j(\lambda') - w_k(\lambda') + \xi(k - j) + \frac{3}{2})\Gamma(w_j(\lambda') \\
 & - w_k(\lambda') + \xi(k - j) - \xi + 2)][\Gamma(w_j(\lambda') - w_k(\lambda') \\
 & + \xi(k - j) + \xi + 1)\Gamma(w_j(\lambda') - w_k(\lambda') + \xi(k - j) + \frac{3}{2})]^{-1}. \tag{B.2}
 \end{aligned}$$

(b) Example for the type I s-s admissible coloured partition.

We consider the type I s-s admissible coloured partition $\mu \in \Lambda_n^{(\gamma)}$ with parity $\rho(\mu) = \overbrace{(1, 1, \dots, 1, 0)}^{\gamma \text{ times}}$. In this case, $I_1(\rho(\mu)) = \{j \in I | j : \text{even}\}$ and $I_2(\rho(\mu)) = \{j \in I | j : \text{odd}\}$.

The formula for Y_{A_μ} is given by

$$Y_{A_\mu} = \xi^{-(\gamma-1)} \prod_{j \in I_1(\rho(\mu))} (p/\gamma + w_j(\mu') + 1/2 - \xi(j - 1))(p/\gamma + w_j(\mu') + 1 - \xi j). \tag{B.3}$$

The explicit form of Y_{B_λ} is given by the same formula in (a) with replacement of $I_1(\rho(\lambda))$ and $I_2(\rho(\lambda))$ by $I_2(\rho(\mu))$ and $I_1(\rho(\mu))$, respectively.

(c) Example for the type II s-s admissible coloured partition Finally, we consider the type II s-s admissible coloured partition $\eta \in \Lambda_n^{(\gamma)}$ with parity

$$\rho(\eta) = (\overbrace{0, 1, 0, 1, \dots, 0, 1, 0}^{(\gamma+3)/2}, \overbrace{0, 1, 0, 1, \dots, 0, 1, 0}^{(\gamma-3)/2}) \quad ((\gamma + 3)/2 : \text{odd}).$$

In this case, $I_1(\rho(\eta)) = \{1, \dots, \beta + 2\}$ and $I_2(\rho(\eta)) = \{\beta + 3, \dots, 2\beta + 1\}$.

We have the explicit formulae for Y_{A_η} and Y_{B_η}

$$\begin{aligned}
 Y_{A_\eta} &= \xi^{-(\gamma+3)} \prod_{\substack{j \in I_1(\rho(\eta)) \\ j: \text{odd}}} (p/\gamma + w_j(\eta') - \xi(j - 1))(p/\gamma + w_j(\eta') + \frac{1}{2} - \xi j) \\
 & \times \prod_{\substack{j \in I_1(\rho(\eta)) \\ j: \text{even}}} (p/\gamma + w_j(\eta') + \frac{1}{2} - \xi(j - 1))(p/\gamma + w_j(\eta') + 1 - \xi j) \tag{B.4}
 \end{aligned}$$

$$\begin{aligned}
 Y_{B_\eta} &= \xi^{\gamma+5} \Gamma((1 + 1/\gamma)/2)^{-\gamma} \prod_{j=1}^{\gamma} \xi^{-2w_j(\eta')} \\
 & \times \prod_{s=1,2} \prod_{\substack{j \in I_s(\rho(\eta)) \\ j: \text{odd}}} \Gamma(w_j(\eta') - \xi(j - 1))\Gamma(w_j(\eta') - \xi j + \frac{1}{2}) \\
 & \times \prod_{s=1,2} \prod_{\substack{j \in I_s(\rho(\eta)) \\ j: \text{even}}} \Gamma(w_j(\eta') - \xi(j - 1) + \frac{1}{2})\Gamma(w_j(\eta') - \xi j + 1)
 \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{s=1,2} \prod_{\substack{j,k \in I_s(\rho(\eta)) \\ j < k \\ j,k: \text{odd}}} [\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j))\Gamma(w_j(\eta') \\
 & - w_k(\eta') + \xi(k-j) - \xi + \frac{1}{2})][\Gamma(w_j(\eta') - w_k(\eta') \\
 & + \xi(k-j) + \xi + \frac{1}{2})\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + 1)]^{-1} \\
 & \times \prod_{s=1,2} \prod_{\substack{j,k \in I_s(\rho(\eta)) \\ j < k \\ j,k: \text{even}}} [\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j))\Gamma(w_j(\eta') \\
 & - w_k(\eta') + \xi(k-j) - \xi + \frac{1}{2})][\Gamma(w_j(\eta') - w_k(\eta') \\
 & + \xi(k-j) + \xi + \frac{1}{2})\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + 1)]^{-1} \\
 & \times \prod_{s=1,2} \prod_{\substack{j,k \in I_s(\rho(\eta)) \\ j < k \\ j: \text{odd}, k: \text{even}}} [\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) - \frac{1}{2})\Gamma(w_j(\eta') \\
 & - w_k(\eta') + \xi(k-j) - \xi)][\Gamma(w_j(\eta') - w_k(\eta') \\
 & + \xi(k-j) + \xi)\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + \frac{1}{2})]^{-1} \\
 & \times \prod_{s=1,2} \prod_{\substack{j,k \in I_s(\rho(\eta)) \\ j < k \\ j: \text{odd}, k: \text{even}}} [\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + \frac{1}{2})\Gamma(w_j(\eta') \\
 & - w_k(\eta') + \xi(k-j) - \xi + 1)][\Gamma(w_j(\eta') - w_k(\eta') \\
 & + \xi(k-j) + \xi + 1)\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + \frac{3}{2})]^{-1} \\
 & \times \prod_{\substack{j \in I_1(\rho(\eta)), k \in I_2(\rho(\eta)) \\ j,k: \text{odd}}} [\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + 1)\Gamma(w_j(\eta') \\
 & - w_k(\eta') + \xi(k-j) - \xi + \frac{3}{2})][\Gamma(w_j(\eta') - w_k(\eta') \\
 & + \xi(k-j) + \xi + \frac{1}{2})\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + 1)]^{-1} \\
 & \times \prod_{\substack{j \in I_1(\rho(\eta)), k \in I_2(\rho(\eta)) \\ j,k: \text{even}}} [\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + 1)\Gamma(w_j(\eta') \\
 & - w_k(\eta') + \xi(k-j) - \xi + \frac{3}{2})][\Gamma(w_j(\eta') - w_k(\eta') \\
 & + \xi(k-j) + \xi + \frac{1}{2})\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + 1)]^{-1} \\
 & \times \prod_{\substack{j \in I_1(\rho(\eta)), k \in I_2(\rho(\eta)) \\ j: \text{odd}, k: \text{even}}} [\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + \frac{1}{2})\Gamma(w_j(\eta') \\
 & - w_k(\eta') + \xi(k-j) - \xi + 1)][\Gamma(w_j(\eta') - w_k(\eta') \\
 & + \xi(k-j) + \xi)\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + \frac{1}{2})]^{-1} \\
 & \times \prod_{\substack{j \in I_1(\rho(\eta)), k \in I_2(\rho(\eta)) \\ j: \text{even}, k: \text{odd}}} [\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + \frac{3}{2})\Gamma(w_j(\eta') \\
 & - w_k(\eta') + \xi(k-j) - \xi + 2)][\Gamma(w_j(\eta') - w_k(\eta') \\
 & + \xi(k-j) + \xi + 1)\Gamma(w_j(\eta') - w_k(\eta') + \xi(k-j) + \frac{3}{2})]^{-1}.
 \end{aligned} \tag{B.5}$$

References

- [1] Ha Z N C and Haldane F D M 1992 *Phys. Rev. B* **46** 9359
- [2] Minahan J A and Polychronakos A P 1993 *Phys. Lett. B* **302** 265
- [3] Calogero F 1969 *J. Math. Phys.* **10** 2191
Calogero F 1969 *J. Math. Phys.* **10** 2197
- [4] Sutherland B 1971 *Phys. Rev. A* **4** 2019
Sutherland B 1972 *Phys. Rev. A* **5** 1372
Sutherland B 1971 *J. Math. Phys.* **12** 246
Sutherland B 1971 *J. Math. Phys.* **12** 251
- [5] Kato Y 1997 *Phys. Rev. Lett.* **78** 3191
- [6] Baker T H and Forrester P J 1997 *Nucl. Phys. B* **492** 682
- [7] Dunkl C F 1998 *Commun. Math. Phys.* **197** 451
- [8] Kato Y and Yamamoto T 1998 *J. Phys. A: Math. Gen.* **31** 9171
(Kato Y and Yamamoto T 1998 *Preprint cond-mat/9806095*)
- [9] Takemura K and Uglov D 1997 *J. Phys. A: Math. Gen.* **30** 3685
- [10] Uglov D 1998 *Commun. Math. Phys.* **191** 663
- [11] Kato Y, Yamamoto T and Arikawa M 1997 *J. Phys. Soc. Japan* **66** 1954
- [12] Kawakami N 1992 *Phys. Rev. B* **46** 1005
Kawakami N 1992 *Phys. Rev. B* **46** 3191
Kawakami N 1993 *J. Phys. Soc. Japan* **62** 2270
Kawakami N 1993 *J. Phys. Soc. Japan* **62** 4163
- [13] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)
- [14] Ha Z N C 1994 *Phys. Rev. Lett.* **73** 1574
Ha Z N C 1995 *Phys. Rev. Lett.* **74** 620 (errata)
Ha Z N C 1995 *Nucl. Phys. B* **435** 604
- [15] Lesage F, Pasquier V and Serban D 1995 *Nucl. Phys. B* **435** 585